Localization in fractal and multifractal media

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The propagation of waves in highly inhomogeneous media is a problem of interest in multiple fields including seismology, acoustics, and electromagnetism. It is also relevant for technological applications such as the design of sound absorbing materials or the fabrication of optically devices for multiwavelength operation. A paradigmatic example of a highly inhomogeneous media is one in which the density or stiffness has fractal or multifractal properties. We investigate wave propagation in one-dimensional media with these features. We have found that, for weak disorder, localization effects do not arrest wave propagation provided that the box fractal dimension D of the density profile is $D \leq 3/2$. This result holds for both fractal and multifractal media providing thus a simple universal characterization for the existence of localization in these systems. Moreover, we show that our model verifies the scaling theory of localization and discuss practical applications of our results.

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Wave propagation[1,2] in disordered or highly inhomogeneous media is a recurrent topic of research in physics. This is not surprising due to its broad range of potential applications. The propagation of electromagnetic waves in highly inhomogeneous fractal media[3–6] was studied experimentally in Refs. 4 and 5. Anderson localization effects in acoustic waves have been observed in experiments[7] and numerical simulations.[8] Seismic waves[9] also propagate in a highly inhomogeneous medium as there is a growing consensus[10,11] that the distribution of fractures and densities in the earth inner structure has a fractal distribution. A detailed understanding of this problem is also relevant for the fabrication of optically active devices for multiwavelength operation,[3] the fabrication of a multiple gap and multiple pass band microstrip resonator filter,[6] the design of sound absorbing materials and the nondestructive characterization of fractured materials.[12]

Rigorous theoretical results are known only for one-dimensional (1D) matter waves in correlated random potentials $V(n)$. Kotani[13] demonstrated that Anderson localization—stop of diffusion and exponential decay of eigenfunctions—occurs for any energy and amount of disorder[1,2] provided that correlations decay as $\langle V(n)V(0)\rangle \propto 1/n^{\alpha}$ with $\alpha > 0$ or faster. Anderson localization effects are also suppressed as the degree of differentiability of the potential increases.[14] For other recent studies of localization in correlated 1D potentials, we refer to Ref. 15. It is unclear to what extent these results still hold in the case of classical waves in a fractal-like media.

The main aim of this Brief Report is to answer this question. We investigate what are the most general circumstances in which a perturbation in a highly heterogeneous 1D medium, described by a fractal or multifractal density, propagates without ever experiencing Anderson localization.

Our main result is that a perturbation will propagate indefinitely in 1D provided that the box fractal dimension of the medium density/stiffness fluctuations is $D \leq 3/2$. These results are still valid if the density profile is multifractal.

Monofractal density. We investigate a discretized version of the 1D scalar wave equation,

$$\rho_n(\psi_{n+1} - \psi_n) - \rho_{n-1}(\psi_n - \psi_{n-1}) + \omega^2 \psi_n = 0, \tag{1}$$

where ω are the eigenfrequencies of the system, $\rho_n = \rho_0(1+\eta_n)$ the ratio between the stiffness and the density of the material, and ρ_0 is the average value. The fluctuations η_n are described by a fractal function with box fractal dimension D. For the sake of simplicity, we study localization properties in a narrow region of frequencies outside the origin $\omega \approx 0$ as a function of the box dimension D and the system size L. We will restrict ourselves to the region of weak disorder. This is a natural choice as our main motivation is to put forward a characterization for the existence of delocalized eigenmodes. As examples of density profiles with a monofractal spatial—labeled by n—distribution, we employ (i) the Weierstrass curve,

$$\eta_n^{we} = \sum_k^{\infty} \cos(\gamma^k n/L + \phi_k)/\gamma^{(2-D)k} \tag{2}$$

with $\gamma > 1$, ϕ_k is a box distributed random number $\in [0, 2\pi]$ and n is a spatial variable. The box fractal dimension is $D^{we} = D$;[16,17] (ii) a fractional Brownian motion (fBm). This is a generalization of the standard Brownian motion characterized by stationary increments—though the process itself is nonstationary—self-similarity, and a variance $\sigma^2 \propto n^{2H}$, where H is the Hurst exponent. We note that, due to its self-similar character, every realization of the process is a fractal curve with box dimension $D = 2 - H$.[16] In both cases in order to have a well-defined continuous limit, we rescale the potential such that the average is $\langle \eta \rangle = 0$. The standard deviation $\sigma_\eta = \sqrt{\langle \eta^2 \rangle}$ controls the strength of disorder. The box-dimension D for a given H is obtained by the method presented in Ref. 18. It basically consists in calculating the average curve length $\langle L(k) \rangle$ over an interval of width k. If $\langle L(k) \rangle \propto k^{-D}$, then the curve is fractal with a box dimension

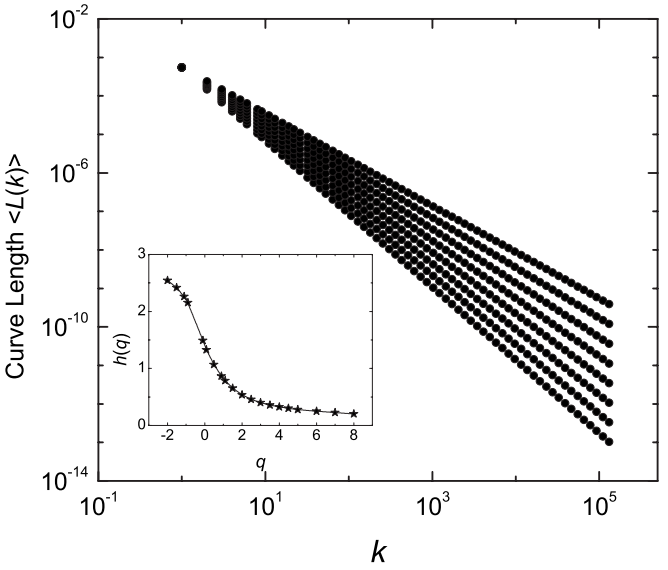

FIG. 1. Average length $\langle L(k)\rangle$ of the medium over an interval of width k. By definition the slope of the curve is the box dimension D. The results correspond to a density profile given by a fBm with Hurst exponent H. H ranges from 0.1 to 0.8, in steps of 0.1, from botton to top. In all cases, $D=2-H$. Inset: generalized Hurst exponent $h(q)[h(2)\equiv H]$ (Ref. 19) that describes the scaling of a multifractal density profile generated by a multiplicative random cascade process (see text).

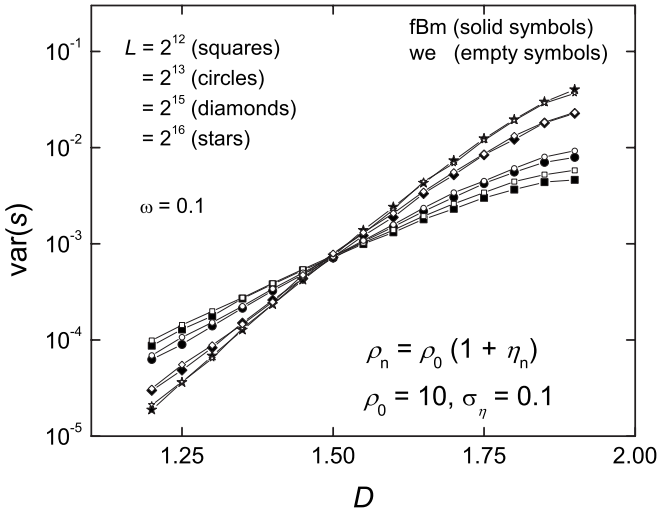

FIG. 2. var(s) as a function of D for $\omega\approx 0.1$, $\sigma_\eta=0.1$, and different sizes L obtained by numerical diagonalization of the wave equation Eq. (1) for a density profile described by the Weiertrass function Eq. (2), and a fractional Brownian motion (see text for details). In both cases, var(s) increases with the system size—a signature of Anderson localization—only for $D>3/2$. This indicates that $D=D_c=3/2$ is the maximum box dimension for which a band of extended modes can occur.

D. In Fig. 1, we show that for the fBm $D_{\mathrm{fBm}}=2-H$. We compute the eigenmodes ω of Eq. (1) by using standard numerical diagonalization techniques. For a given disorder σ_η and frequency window, the number of eigenmodes obtained is at least 2×10^5. In order to determine the importance of Anderson localization effects, we carry out a finite-size scaling analysis of the spectrum.[20,21] The variance var(s) of the eigenmode spacing distribution $P(s)$ is chosen as the scaling variable.[21] $P(s)$ is the probability of finding two neighboring eigenmodes at a distance $s=(\omega_{i+1}-\omega_i)/\Delta$ and

$$\mathrm{var}(s)\equiv\langle s^2\rangle-\langle s\rangle^2=\int_0^\infty dss^2P(s)-1, \tag{3}$$

where $\langle\cdots\rangle$ denotes frequency and ensemble averaging and Δ is the local mean eigenmode spacing. In case that propagation is not stopped by localization effects, var(s)$\approx$0.273 for diffusive motion and var(s)=0 for ballistic motion. By contrast var(s)=1 indicates that a perturbation cannot propagate indefinitely in the medium. A value of var(s) that increases (decreases) with system size signals that the perturbation will eventually (never) get localized.[21]

The results for var(s) for the fBm and the Weiertrass density profile, at fixed ω and disorder are shown in Fig. 2 as a function of D. For $D>3/2$, the variance var(s) increases with the system size. This indicates stop of wave propagation. By contrast for $D<3/2$, var(s) decreases with the system size thus suggesting that, in this case, localization never occurs. Therefore, in fractal media the transport properties for sufficiently weak disorder are to a great extent controlled by the fractal box dimension D of the density/stiffness. A natural question to ask is the degree of universality of this characterization in terms of D. In order to answer this question, we investigate density profiles with multifractal properties. In this case, the box dimension can still be defined but the scaling properties of the density profile depend on an infinite number of multifractal dimensions.[16] It is thus unclear whether other fractal dimensions control localization related effects.

Multifractal density generated by a cascade process. We first study the effect of multifractality in a density profile whose two-point correlation is $\langle\eta_n\eta_0\rangle\propto\delta(n)$. Our motivation is to estimate the quantitative effect of a multifractal media by comparing it with one, an uncorrelated Gaussian noise, which is not multifractal but have the same two-point correlation function. The multifractal density fluctuations are generated iteratively by a variant of a multiplicative random cascade process.[17] The length of the process doubles in each iteration. Initially the density consists just of one value $\eta^{(0)}=\eta_0$. In the next generation, it has two values $\eta^{(1)}=m_1^{(1)}\eta_0$ and $\eta^{(2)}=m_2^{(1)}\eta_0$, where m_i are Gaussian random numbers of zero mean and unit variance. After k iterations, the density will have 2^k different values given by $\eta_{2l-1}^{(k)}=\eta_l^{(k-1)}m_{2l-1}^{(k)}$ and $\eta_{2l}^{(k)}=\eta_l^{(k-1)}m_{2l}^{(k)}$. Following Ref. 19, we investigate the scaling properties of these data by using the multifractal detrended fluctuation analysis. The resulting density is characterized by a generalized Hurst exponent $h(q)$ (see Ref. 19 for a definition and Fig. 1). For a monofractal curve, $h(q)=H$ is independent of q. For this multifractal distribution, the box dimension is $D=2$ as that of a Gaussian uncorrelated disorder. According to our previous characterization, localization should occur for any frequency and amount of disorder. A finite-size scaling analysis (not shown), as the one performed in the monofractal case, con-

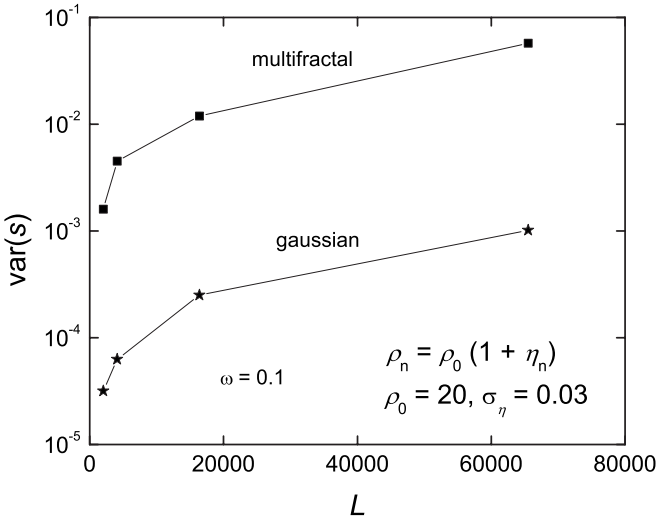

FIG. 3. var(s), as a function of the system size L, fixed disorder σ_η=0.03 and $\omega \approx 0.1$ for a density profile described by a Gaussian white noise and a multifractal one generated by a multiplicative random cascade process (see text). In both cases, the box dimension is D=2 and the two-point correlation function $\langle \eta_n \eta_0 \rangle \propto \delta(n)$. Since $D > 3/2$, all modes are localized. However, localization effects are stronger [var(s) is larger] for the medium whose density is multifractal.

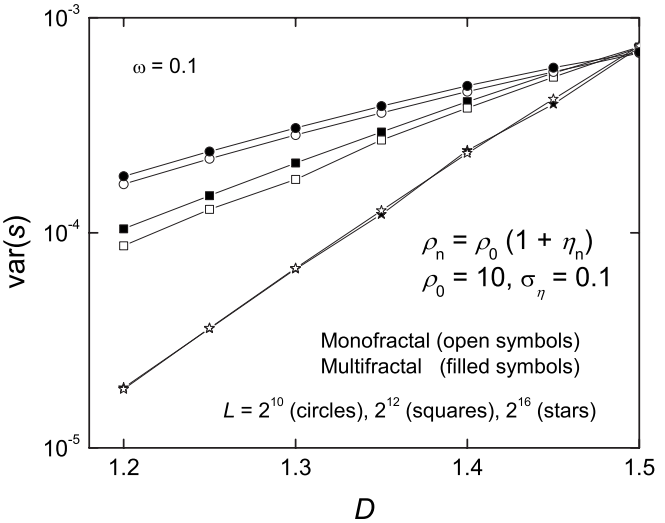

FIG. 4. var(s) as a function of $D < 3/2$ for $\omega \approx 0.1$, σ_η=0.1, and different L for a density profile described by a multifractal random walk with λ^2=0.1, L_0=2^{10}, and a monofractal with the same box dimension D. Multifractality enhaces localization effects but the critical box dimension is still D=D_c=3/2.

firms this prediction. This result might suggest that localization effects in both media are similar. However, this is not the case. Multifractal corrections, which arise in higher-order correlation functions, play an important role. In order to illustrate this, we compare in Fig. 3 var(s) for both models as a function of L at fixed ω and σ_η. We observe that (a) in both models, var(s) increases with the system size. This is a signature of localized modes; (b) multifractal scaling enhances localization effects since var(s) is two orders of magnitude larger in this case. It is unclear to what extent this behavior is universal. Multifractality leads to a broad range of scaling patterns. Therefore, we cannot discard that in certain cases a suppression of localization effects can be observed. From our point of view, the main conclusion of this analysis is that, in order to attain a good understanding of the transport properties in a highly inhomogeneous medium, is very important to know in detail the full distribution function that describes the medium density and not only its first moments.

Multifractal density generated by a generalized random walk. We now move to the case of a multifractal density in which $h(2)$ can be tuned such that the box dimension D changes. Our motivation here is to test whether, as in the monofractal case, absence of localization is only observed for $D < 3/2$. For this purpose, we study the multifractal random walk introduced in Ref. 22. The strength of disorder is set as in the monofractal case. Here we just state the definition of the model and refer to Ref. 22 for details,

$$\eta_n^{mul} = \sum_{i=1}^{n} \eta_i^{id} e^{r(i)}, \tag{4}$$

where η^{id} is a fractional Gaussian noise, $r(i)$ is a Gaussian correlated noise with $\langle r^2 \rangle = \lambda^2 \ln(L_0)$ and $\langle r(i) r(j) \rangle = \lambda^2 \ln f(|i-j|)$ with λ a free parameter describing the strength of the multifractal scaling, L_0 is the largest scale for which multifractal scaling is observed and $f(|i-j|) = \frac{L_0}{1+|i-j|}$ for $|i-j| \leq L_0 - 1$ and the unity otherwise. With these definitions, the resulting fluctuations of the density profile η^{mul} are multifractal with a set of multifractal dimensions that depends on λ. As in the monofractal case, we have carried out a finite-size scaling analysis of the spectrum. The results for var(s) (see Fig. 4), indicate that propagation is arrested only for $D > 3/2$ since in this region var(s) decreases with the system size. Therefore, our characterization for the existence of extended modes is still valid for media with a multifractal density. This together with the previous results on monofractal media strongly suggests that the value of the box dimension D provides a characterization for the absence of Anderson localization in 1D media.

Finally, we investigate whether our findings are consistent with the one parameter scaling theory.[23] According to this theory, var(s) can be expressed by a one-parameter scaling function var(s)=$f[L/\xi(D)]$, where the scaling parameter ξ is the localization length in the localized regime and the correlation length in the extended regime. This relation implies that in a log-log plot of var(s) versus L all data should collapse in a common curve when translated by an amount $\ln \xi(D)$ along the horizontal axis. This curve has a single branch when there is no transition while it develops two separate branches when a transition is present. In Fig. 5, we plot var(s) as a function of L for different values of D at a fixed ω. The obtained result is fully consistent with the scaling hypothesis for var(s). Therefore, the one-parameter scaling theory is still valid for this type of systems.

Applications. One of the main motivations to study wave transport in fractal and multifractal media is its relevance in many realistic situations. We now address some of these applications. The distribution of fractures and densities in the earth inner structure is believed to be fractal/multifractal.[10,11]

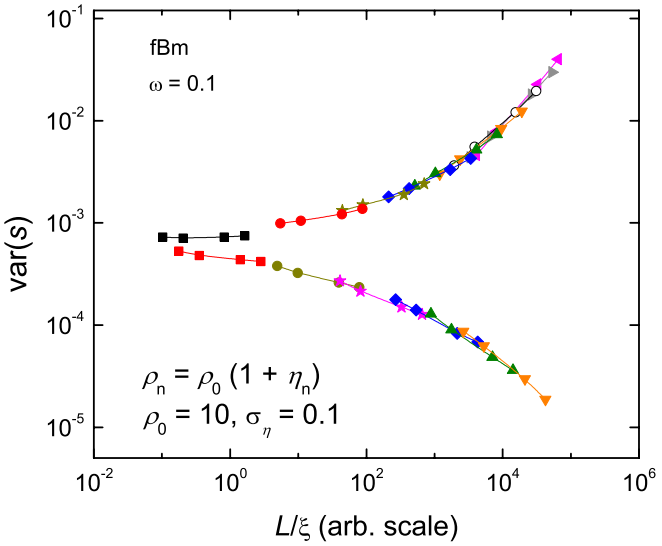

FIG. 5. (Color online) var(s) as a function of the system size for $\omega \approx 0.1$ and $\sigma_\eta = 0.1$ for a density described by a fBm. Each color/symbol corresponds to a different D. The observed data collapse is fully consistent with the one-parameter scaling assumption for var(s) (Ref. 23) (see text).

Therefore, a more comprehensive understanding of the transport properties in such a medium would not only help characterize the earth inner structure but also would be relevant for practical problems such as the minimization of earthquakes effects and the detection of possible large-scale reservoir of oil and gas.[24] We note that our results provide a characterization of the conditions that a fractal medium must meet in order that a wave can propagate through it.

A detailed understanding of the propagation of electromagnetic waves in a highly inhomogeneous medium is a key ingredient in the fabrication of different devices[3,6] (see introduction). In this case, our characterization would be useful to optimize its capabilities. Likewise a comprehensive theory of the attenuation of acoustic waves in a fractal porous medium would improve dramatically the design of sound absorbing materials.

The study of matter waves in cold atoms settings could also benefit from these findings. In this case, the system is a tight-binding Anderson model with a fractal-like potential. This type of potentials might be realized experimentally by using a holographic mask combined with spatial light modulators.[25]

To conclude, we have studied wave propagation in a 1D fractal media. Our main results are (a) the fractal dimension of the potential D controls the strength of localization effects; (b) only for $D < 3/2$ wave propagation is not arrested by localization effects; (c) multifractal corrections do not modify this result but have a profound impact in the transport properties; (d) scaling theory still applies in these systems.

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